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DISTRIBUTION OF STRESSES IN ELASTIC STRONGLY ANISOTROPIC MATERIAL

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Two well-known problems in elasticity theory are considered in this work (stress distribution within an elliptical region, and the contact problem for the half plane) from the point of view of the effect of marked anisotropy. We recall that a material is called markedly anisotropic if Young's modulus in a given direction is much greater than in the orthogonal direction. In particular, the limiting case is studied when the material at the limit is unstretchable. It is discovered that the calculation problem for the constructions mentioned above has a series of specific features.

1. Stress Distribution in an Elliptical Region. A method is suggested in [1, 2] for calculating orthotropic plates based on the fact that for many materials the ratio of complex parameters is small. This method is also based on expanding complex potentials $\Phi_k (k = 1, 2)$ into a series for non-negative powers of the small parameter. It shown below on the example of an explicit solution of the boundary problem for an elliptical region that expansion into a series of one of the complex potentials commences with a negative power of the small parameter and also that the solution of the limiting problem has a feature of the square root type at points of tangency for characteristics of the limiting equation boundary.

We take the generalized Hooke's law for an orthotropic material in the form

$$\sigma_{11} = c_{11}u_{1x_1} + c_{12}u_{2x_2},\ \sigma_{22} = c_{12}u_{1x_1} + c_{22}u_{2x_2},\ \sigma_{12} = c_{66}(u_{1x_2} + u_{2x_1}),$$

where u_1, u_2 are displacements; c_{ij} are elasticity coefficients. We introduce dimensionless stresses and stiffness assuming that

$$d_{ij} = c_{ij}c_{66}^{-1},\ \sigma_{ij}^* = \sigma_{ij}c_{66}^{-1},\ i, j = 1, 2$$

(from here on, for dimensionless stresses the previous notation is retained).

We consider the situation when $d_{22} \gg 1$. This situation corresponds to a composite material reinforced with a single family of very stiff fibers parallel to axis x_2. We assume that $d_{22} = \varepsilon^{-2}$, $\varepsilon \ll 1$. The equation for the function of stresses in the absence of volumetric forces has the form

$$\gamma_1^2 w^\varepsilon_{,x_1^4} + (1 - c\varepsilon^2\gamma_1^2)w^\varepsilon_{,x_1^2x_2^2} + \varepsilon^2 w^\varepsilon_{,x_2^4} = 0 \tag{1.1}$$

($\gamma_1 = d_{11}{}^{-1}$, $c = d_{12}{}^2 + 2d_{12}$). With $\varepsilon = 0$ it is converted into an equation

$$\gamma_1^2 w^0_{,x_2^4} + w^0_{,x_1^2x_2^2} = 0$$

of the composite type with one double family of characteristics x_1 = const. The change in type of equation at the limit makes it possible to expect that the solution of the limiting problem will have features which are absent in anisotropic elasticity. In fact, as was shown previously [3], close to the characteristic part of the boundary occurrence of a boundary layer is observed and the number of boundary conditions at the limit reduced to one.

Accurate solution of the elasticity theory problem for a solid ellipse with stress vector prescribed at the boundary is given in [4]. Using the notation in [4] we write the boundary conditions:

$$2\mathrm{Re}[\Phi_1(z_1) + \Phi_2(z_2)]|_{\partial\Omega} = g_1(\theta),$$
$$2\mathrm{Re}[\mu_1\Phi_1(z_1) + \mu_2\Phi_2(z_2)]|_{\partial\Omega} = g_2(\theta).$$

It is assumed that g_1, g_2 are expanded into a Fourier series:

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$$g_1(\theta) = \alpha_0 + \sum_{k=1}^{\infty} (\alpha_k e^{ik\theta} + \bar{\alpha}_k e^{-ik\theta}),$$

$$g_2(\theta) = \beta_0 + \sum_{k=1}^{\infty} (\beta_k e^{ik\theta} + \bar{\beta}_k e^{-ik\theta}).$$

The equation for the boundary is taken in parametric form $x_1 = a\cos\theta$, $x_2 = b\sin\theta$. For plate equilibrium it is necessary to fulfill the condition $-i(\alpha_1 - \bar{\alpha}_1)b^{-1} = a^{-1}(\beta_1 + \bar{\beta}_1)$. Functions Φ_1 and Φ_2 are expanded into a series with respect to Faber polynomials:

$$\Phi_1 = A_0 + A_1 z_1 + \sum_{k=2}^{\infty} A_k P_{1k}(z_1),\ \Phi_2 = B_0 + B_1 z_2 + \sum_{k=2}^{\infty} B_k P_{2k}(z_2).$$

Here

$$P_{sk}(z_s) = (-1)(a - i\mu_k b)^{-1}\,[(z_s + (z_s - a^2 - \mu_s b^2)^{1/2}) + + (z_s - (z_s - a^2 - \mu_s^2 b^2)^{1/2})^k],\quad s = 1, 2.$$

We assume that $\mu_k = il_k$; l_1 and l_2 are positive roots of the equation

$$(d_{11} - \lambda^2)(1 - \varepsilon^{-2}\lambda^2) + c^2\lambda^2 = 0.$$

Constants A_k, B_k are determined from the set of equations

$$A_k + B_k + \bar{A}_k t_1^k + \bar{B}_k t_2^k = -\alpha_k,\quad A_k\mu_1 + B_k\mu_2 + \bar{A}_k\bar{\mu}_1 t_1^k + \bar{B}_k\bar{\mu}_2 t_2^k = -\beta_k,$$

where $t_k = (a + i\mu_k b)(a - i\mu_k b)^{-1}$ $(k = 1, 2)$. In future we assume that equilibrium conditions are fulfilled and that A_0, $B_0 = 0$.

We consider passage to the limit with $\varepsilon \to 0$. Then l_1 tends towards zero and l_2 tends towards $d = \sqrt{d_{11}}$. More accurately,

$$l_1 = \varepsilon + O(\varepsilon^3),\ l_2 = d + O(\varepsilon^2).$$

We assume that

$$A_k = a_k^\varepsilon + ib_k^\varepsilon,\ B_k = c_k^\varepsilon + id_k^\varepsilon,$$
$$\delta_1 = l_2(1 - t_2^k)(1 + t_1^k) - \varepsilon(1 - t_1^k)(1 + t_2^k),$$
$$\delta_2 = l_2(1 - t_1^k)(1 + t_2^k) - \varepsilon(1 + t_1^k)(1 - t_2^k).$$

Then

$$a_k^\varepsilon = \delta_1^{-1}[(1 + t_2^k)\,\mathrm{Im}\,\beta_k - l_2(1 - t_2^k)\,\mathrm{Re}\,\alpha_k],$$
$$c_k^\varepsilon = \delta_1^{-1}[\varepsilon(1 - t_1^k)\,\mathrm{Re}\,\alpha_k - (1 + t_1^k)\,\mathrm{Im}\,\alpha_k],$$
$$b_k^\varepsilon = -\delta_2^{-1}[(1 + t_2^k)\,\mathrm{Im}\alpha_k + (1 - t_2^k)\,\mathrm{Re}\,\beta_k],$$
$$d_k^\varepsilon = \delta_2^{-1}[(1 - t_1^k)\,\mathrm{Re}\,\beta_k + \varepsilon(1 + t_1^k)\,\mathrm{Im}\,\alpha_k].$$

It is easy to see that a_k^ε, c_k^ε, d_k^ε have a fine limit with $\varepsilon \to 0$. However, b_k^ε is the order $O(\varepsilon^{-1})$. We assume that $b_k^\varepsilon = -\varepsilon^{-1}m^\varepsilon$. Then m^ε converges with $\varepsilon \to 0$ to the value m^0:

$$m^0 = \delta_3^{-1}[d(1 + t_2^k)\,\mathrm{Im}\,\alpha_k + (1 - t_2^k)\,\mathrm{Re}\,\beta_k],$$
$$\delta_3 = 2ba^{-1}kd(1 + t_2^k) - 2(1 - t_2^k).$$

Consequently, asymptotic expansion $\Phi_1(x_1 + i\varepsilon x_2)$ commences with the power ε^{-1}; therefore expansion with respect to powers ε, suggested in [1], is incorrect. In fact

$$\Phi_1 = \sum_{k=2}^{\infty} (a_k^\varepsilon + ib_k^\varepsilon)[\mathrm{Re}P_{1k}(z_1) + i\mathrm{Im}P_{1k}(z_1)].$$

Then

$$\operatorname{Re}\Phi_1 = \sum^{\infty}(a_k^\varepsilon \operatorname{Re} P_{1k}(z_1) - b_k^\varepsilon \operatorname{Im} P_{1k}(z_1)),$$

$$\operatorname{Im}\Phi_1 = \sum_{k=2}^{\infty}(b_k^\varepsilon \operatorname{Re} P_{1k}(z_1) + a_k^\varepsilon \operatorname{Im} P_{1k}(z_1)). \quad (1.2)$$

It follows from (1.2) that $\operatorname{Re}\Phi_1 = O(1)$, $\operatorname{Im}\Phi_1 = O(\varepsilon^{-1})$. We assume that $P_{ik}(x) = -2\cos kt$, $t = \arccos x/a$, then

$$P_{1k}(z_1) = P_{1k}(x) + i\varepsilon y P'_{1k}(x) + O(\varepsilon^2).$$

Here $P_{1k}'(x) = -2ka^{-1}\sin kt(\sin t)^{-1}$.

The preceding calculations make it possible to complete a passage to the limit in Eqs. (1.2). If w^0 is the solution of the limiting problem, then

$$w^0_{,x_1} = x_2\varphi_1'(x_1) + \varphi_2'(x_1) + \operatorname{Re}\Phi_2'(z_3),\ z_3 = x_1 + idx_2,$$

$$w^0_{,x_2} = \varphi_1'(x_1) + \operatorname{Re} id\Phi_2'(z_3),$$

where

$$\varphi_1(x_1) = \sum_{k=2}^{\infty} a_k \cos(k \arccos x_1/a);$$

$\varphi_2''(x_1)$ is the first term in Eq. (1.2). It is easy to prove that the boundary conditions for the original problem are fulfilled at the limit. We calculate stress $\sigma_{22} = w_{x_1x_1}$ in the limiting problem:

$$\sigma_{22} = x_2\varphi''_1(x_1) + \varphi''_2(x_1) + \operatorname{Re}\Phi''_2(z_3),$$

$$\varphi_1(x_1) = -\sum_{k=2}^{\infty} a_k(k\cos kt(\sin t)^{-2} - \sin kt\cos kt(\sin t)^{-3}).$$

It is apparent that the product $x_2\varphi_1''(x_1)$ has the feature $(x^2 - a^2)^{-1/2}$ with $x = +a$, $x = -a$. It is easy to see that in the limiting strain problem $\varepsilon_{22} = 0$ and the material is unstretchable in the direction of axis x_2. At the limit the rule of state has the form

$$\sigma_{11} = d_{11}u_{1,x_1},\ \sigma_{12} = u_{1,x_2} + u_{2,x_1},\ \sigma_{22} = q + d_{12}u_{1,x_1},\ \varepsilon_{22} = 0.$$

Here $q(x_1, x_2)$ is a new unknown function (reaction of the material to limited unstretchability). Displacements u_1, u_2, and function q satisfy the set of equations

$$d_{11}u_{1,x_1x_1} + u_{1,x_2x_2} = 0,\ u_{2,x_2} = 0,$$

$$q_{x_2} + u_{2,x_1x_1} + (1 + d_{12})u_{1,x_1x_2} = 0,$$

with the general solution

$$u_1 = \operatorname{Re}\Phi(x_1 + idx_2),\ u_2 = u_2(x_1),$$

$$q = -x_2u_{2,x_1x_1} - (1 + d_{12})\, u_{1,x_1} + \omega(x_1) \quad (1.3)$$

and it depends on three unknown functions: ω, Φ, u_2. It is noted that close to the point $x_1 = a$, $-a$ a so-called "free" boundary layer arise and in order to construct a uniform solution everywhere in a closed region for the original problem to the limiting solution it is necessary to add boundary layer function.

We give one more example demonstrating the incorrectness of the direct passage to the limit. The potential of a simple layer in a bounded singly-connected region Q with boundaries class C^2 may be described in the form [5]

$$u_1 = \pi^{-1}\operatorname{Im}\oint_{\partial Q}\{[A_1\ln\sigma_1 + A_2\ln\sigma_2]m_1 + [B_1\ln\sigma_1 + B_2\ln\sigma_2]m_2\}ds,$$

$$u_2 = \pi^{-1}\operatorname{Im}\oint_{\partial Q}\{[B_1\ln\sigma_1 + B_2\ln\sigma_2]m_1 + [C_1\ln\sigma_1 + C_2\ln\sigma_2]m_2\}ds, \quad (1.4)$$

where m_1, m_2 are unknown densities; $\sigma_1 = x_1 - \xi_1 + i\varepsilon(x_2 - \xi_2)$; $\sigma_2 = x_1 - \xi_1 + il_2(x_2 - \xi_2)$; $(\xi_1, \xi_2) \in \partial Q$; coefficients A_k, B_k, $C_k (k = 1, 2)$ have the following order with respect to ε:

$$A_1 = i(1 + d_{12})^2 d_{11}^{-1}(\varepsilon^2 - d_{11})^{-1}\varepsilon^3,\ A_2 = id_{11}^{-1/2},$$
$$B_1 = \varepsilon^2(1 + d_{12})(\varepsilon^2 - d_{11})^{-1},\ B_2 = (1 + d_{12})d_{11}^{-1}\varepsilon^2,$$
$$C_1 = i\varepsilon,\ C_2 = i(1 + d_{12})^2 d_{11}^{-1/2}(\varepsilon^2 - d_{11})^{-1}\varepsilon^4.$$

We recall that the potential of a simple layer resolves the first main problem of elasticity theory (at the boundary the force vector is prescribed) and it has a logarithmic growth at infinity. If in (1.4) we pass to the limit with $\varepsilon \to 0$ we obtain $u_2 = 0$. It is evident that this solution does not correspond to the general solution (1.3). If the solution obtained above is analyzed for a solid ellipse, then it should be assumed that product $\varepsilon m_2(s) = \Phi_2(s)$ has a finite limit with $\varepsilon \to 0$, and then

$$u_1 = (\pi d)^{-1} \int_{\partial Q} m_1(s) \ln r_2 ds,\ u_1 = \pi^{-1} \int_{\partial Q^*} \Phi_2(s) \ln |x_1 - x_1(s)| ds,$$
$$r_2 = ((x_1 - x_1(s))^2 + d(x_2 - x_2(s))^2)^{1/2}$$

(∂Q^* is the uncharacteristic part of the boundary).

2. Rigid Die with an Elastic Orthotropic Half Plane. This problem has been studied effectively and in the future we are interested in the main limiting case when the half plane is unstretchable in the direction of axis x_2. In the absence of friction the boundary conditions of the problem are written as:

$$\sigma_{22} = 0 \text{ with } x_2 = -0,\ |x_1| \geq a,\ u_2 = f(x_1) \text{ with } |x_1| \leq a; \tag{2.1}$$

$$\sigma_{12} = 0 \text{ with } x_2 = -0. \tag{2.2}$$

The interval $(-a, a)$ is the possible region of contact. In the notation in [4] stresses σ_{12}, σ_{22}, and displacement u_2 are expressed in terms of Lekhnitskii complex potential as follows:

$$\sigma_{22} = 2\mathrm{Re}[\Phi_1(z_1) + \Phi_2(z_2)],\ \sigma_{12} = -2\mathrm{Re}[\mu_1\Phi_1(z_1) + \mu_2\Phi_2(z_2)],$$
$$u_2 = 2\mathrm{Re}[q_1\Phi_1(z_1) + q_2\Phi_2(z_2)],\ q_k = a_{12}\mu_k + a_{22}\mu_k^{-1},\ k = 1, 2.$$

First we provide the solution of the original problem in a form convenient for further analysis. We assume that

$$\Phi_1(z_1) = (\pi i)^{-1}\mu_2(\mu_1 - \mu_2)^{-1} \int_{-a}^{a} p(x) \ln(x - z_1)dx,$$

$$\Phi_2(z_2) = -(\pi i)^{-1}\mu_1(\mu_1 - \mu_2)^{-1} \int_{-a}^{a} p(x) \ln(x - z_2)dx.$$

Then boundary condition (2.2) is fulfilled in the same way and in order to determine unknown density p(x) we have an integral equation

$$m \int_{-a}^{a} p(t) \ln|t - x| dt = f(x),\ m = \mathrm{Re}\, i(\mu_1 q_2 - \mu_1 q_2)(\mu_1 - \mu_2)^{-1}\pi^{-1}$$

with the solution

$$p(x) = (\pi m)^{-1}(a^2 - x^2)^{-1/2}[\pi^{-1} \int_{-a}^{a} f(t)(a^2 - t^2)^{-1/2}(t - x)^{-1}dt - \pi P]$$
$$= g(x)m^{-1}.$$

The preceding equations make it possible to determine entirely the stress-strained state in the lower half plane $x_2 \leq 0$.

As above, we consider the limiting situation when $\varepsilon \to 0$. At the limit we obtain (as previously an upper zero index signifies the limit of a value with $\varepsilon \to 0$)

$$u_2^0(x_1) = -\int_{-a}^{a} g(t) \ln|t - x_1| dt,$$

$$u_1^0(x_1, x_2) = -\mathrm{Re}(\pi d)^{-1} \int_{-a}^{a} g(t) \ln(t - z_3)dt,\ z_3 = x_1 + idx_2,$$

$$\sigma_{22}^0 = x_2\pi^{-1}\int_{-a}^{a} g(t)(t - x_1)^{-2}dt - \mathrm{Re}(\pi i d)^{-1}\int_{-a}^{a} g(t)(t - z_3)^{-1}dt,$$

$$\sigma_{22}(x_1, -0) = (\pi d)^{-1}g(x).$$

The first integral in the equation for $\sigma_{22}{}^0$ is hypersingular and it only exists in the sense of a Hadamard finite part. It is noted that $u_2{}^0 \approx C_1\ln|x_1|$ with large $|x_1|$, $u_1(x_1, x_2) \approx C_2\ln r_2$ with large r_2; here $u_2{}^0$ only depends on x_1, and $\sigma_{22}{}^0$ increases linearly with respect to x_2; nonetheless, pressure beneath the die is finite.

In order to illustrate the solutions obtained above we consider two particular cases.

A. A Die with a Flat Base. We assume that $f(x_1) = c$. Then

$$u_2^0(x_1) = c \quad \text{with} \quad |x_1| \leq a,$$

$$u_2^0 = -P[\ln(x_1 + (x_1^2 - a^2)^{1/2}) - \ln a] + c \quad \text{with} \quad x_1 > a,$$

$$u_2^0 = P[\ln(x_1 + (x_1^2 - a^2)^{1/2}) - \ln a] + c \quad \text{with} \quad x_1 < -a,$$

$$\sigma_{22}^0 = -x_2 u_{2x_2x_2} - \mathrm{Re}P(id(z_3^2 - a^2)^{1/2})^{-1}.$$

If the force pressing against the die is P_0, then $P_0 = P\pi d^{-1}$ and pressure beneath the die $p(x_1)$ is given by the classical equation

$$p(x_1) = P_0\pi^{-1}(a^2 - x_1^2)^{-1/2}.$$

It is noted that stress $\sigma_{22}{}^0$ on the extension of the die has the feature $(x_1{}^2$ N $a^2)^{-3/2}$, which is absent in the classical problem. The resons for this are explained below.

B. A Die with a Rounded Base. We assume that as always $f(x_1) = x_1{}^2(2R)^{-1}$, where R is radius of curvature. Then

$$u_2^0(x_1) = x_1^2(2R)^{-1} \quad \text{with} \quad |x_1| \leq a,$$

$$u_2(x_1) = -P[\ln(x_1 + (x_1^2 - a^2)^{1/2}) - \ln a] + x_1^2(2R)^{-1} - x_1(2R)^{-1}(x_1^2 - a^2)^{1/2}$$

where $x_1 > a$, and a similar equation occurs with $x_1 < -a$,

$$\sigma_{22}^0(x_1, -0) = -d^{-1}(a^2 - x_1^2)^{-1/2}[-P + R^{-1}(x_1^2 - 2^{-1}a^2)], \quad |x_1| \leq a.$$

The solution is physically possible if pressure beneath the die is not negative, i.e., if $P \geq a^2(2R)^{-1}$. If P does not satisfy this condition it means that force P is insufficient in order to be in complete contact with an elastic body. As is normal [6, 7], in this case in order to determine the contact region it is necessary to use the requirement for reversion to zero of the contact pressure at the ends of the contact region. We point out that as in the previous case $\sigma_{22}{}^0$ with $|x| > a$ has the feature $(x^2 - a^2)^{-3/2}$.

Appearance of this feature for stress $\sigma_{22}{}^0$ is connected with the fact that with $\varepsilon \ll 1$ the original problem is singularly disturbed and close to point $x_1 = \pm a$ a free boundary layer arises. Here only a zero term of the asymptotic is constructed; in order to construct a uniform asymptotic it is necessary to add a boundary layer function to this solution.

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